A Note for Practitioners: Stability and Instability Investgations by the Sign Change of the Lyapunov Exponent of the Black-Scholes Formula of an European Call Option

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May 17, 2011

Abstract

In this paper the stability of the Black-Scholes Formula of a European call option is investigated. The value of the call option is mainly instable respectively chaotic. For short times the value function can be in the stability region. This depends strongly on the volatility and the interest rate. Some finance experts have the doctrine, that call options can be traded over a few decades without high risks. This will be falsified in this note.
The other aim of this paper is to test the method of Lyapunov exponent in a financial problem and to extend the scope of econophysics.

1 Introduction

The Black-Scholes Formulae describe the price of a stock option. In this paper I concentrate on Europen call options. The Black-Scholes Formulae are solutions of the Black-Scholes partial differential equation, which is defining the mathematical model. For details see [1] and [2].
There is yet no article on the investigation, when the value of a European call option depending on the time t becomes instable by means of the Lyapunov exponent.

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2 Black-Scholes Formula for a European Call Option

The Black-Scholes Formula for European call option reads [2] or [1]:

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)} \tag{1}$$

where $C(S,t)$ is the value the call option, depending on time t (in years) and the spot price S of the underlying stock. r is the constant interest rate, K is the strike price of the option and T the expiry time of the option.
$N(x)$ is the cumulant distribution function of the standard normal distribution:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz \tag{2}$$

The quantities d_1 and d_2 are given as:

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{(T-t)}} \tag{3}$$

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{(T-t)}} \tag{4}$$

3 Searching for instability regions and stability regions in the Black-Scholes Formula for a European call option by the sign change of the Lyapunov Exponent

3.1 Derivation of an Inequality

At first I reformulate eq. (1) for the value of an European call option. Time t (in years) is replaced by the natural number i.

$$C_i = N(d_{1,i})S - N(d_{2,i})Ke^{-r(T-i)} \tag{5}$$

And C_{i+1} reads:

$$C_{i+1} = N(d_{1,i+1})S - N(d_{2,i+1})K \underbrace{e^{-r(T-(i+1))}}_{Ke^{-r(T-i)}e^{r}} \tag{6}$$

Eq. (5) yields:

$$\frac{N(d_{1,i})S - C_i}{N(d_{2,i})} = Ke^{-r(T-i)} \tag{7}$$

Eq.(7) is inserted into eq.(6). This leads to the following expression:

$$C_{i+1} = N(d_{1,i+1})S + \frac{N(d_{2,i+1})}{N(d_{2,i})}e^r \cdot C_i - \frac{N(d_{2,i+1})N(d_{1,i})S}{N(d_{2,i})} \cdot e^r \tag{8}$$

This expression is useful in order to calculate the Lyapunov exponent λ. The Lyapunov exponent of the map $x_{n+1} = f(x_n)$ reads [3], p.24f:

$$\lambda(x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log(|f'(x_i)|) \tag{9}$$

$x_{n+1} = f(x_n)$ is here $C_{i+1} = f(C_i)$. For dynamical instability ($\lambda > 0$) the argument of the logarithm has to be:

$$\left|\frac{\partial C_{i+1}}{\partial C_i}\right| = \left|\frac{N(d_{2,i+1})}{N(d_2, i)}e^r\right| > 1 \tag{10}$$

Eq.(10) rewritten:

$$|N(d_{2,i+1})e^r| > |N(d_2, i)| \tag{11}$$

On both sides the inetgral $\int\limits_{-\infty}^{0} e^{-\frac{z^2}{2}} dz$ is subtracted[1]:

$$\left(\int\limits_{-\infty}^{d_{2,i+1}} e^{-\frac{z^2}{2}} dz\right) e^r > \int\limits_{-\infty}^{d_{2,i}} e^{-\frac{z^2}{2}} dz \Bigg| - \int\limits_{-\infty}^{0} e^{-\frac{z^2}{2}} dz \tag{12}$$

Replace $-\frac{z^2}{2}$ by $-t^2$. Then it holds $t = \frac{z}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}dz = dt$

$$\left(\int\limits_{0}^{\frac{d_{2,i+1}}{\sqrt{2}}} e^{-t^2} dt\right) e^r > \int\limits_{0}^{\frac{d_{2,i}}{\sqrt{2}}} e^{-t^2} dt \tag{13}$$

[1]The absolute sign is not anymore necessary.

The Gaussian error function is defined as follows and can be written as infinte series ([4] 8.250 1 (Definition of $erf(x)$) and 8.253 1. 2nd expression):

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!} \quad (14)$$

The instability inequality eq. (13) assumes the following form (>0 means instability):

$$F = F(i; r, \sigma, T, K, S) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)(k-1)!} \left(\left(\frac{d_{2,i+1}}{\sqrt{2}} \right)^{2k-1} \cdot e^r - \left(\frac{d_{2,i}}{\sqrt{2}} \right)^{2k-1} \right) > 0 \quad (15)$$

The instability function F is positve for the case of instability. $F < 0$ means stability. F depends on the discrete time i (in years), the expiry time T of the call option, the volatility σ, the interest rate r, the exercise price (i.e. strike price) of the option and the spot price S of the underlying stock. The definitions for $d_{2,i+1}$ and $d_{2,i}$ (eq.(3) and eq.(4)) are inserted:

$$F(i; T, \sigma, r, K, S) = F =: \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)(k-1)!} \left(\underbrace{\left(\frac{\ln(\frac{S}{K} + (r - \frac{\sigma^2}{2}(T-(i+1)))}{\sqrt{2}\sigma\sqrt{T-(i+1)}} \right)^{2k-1}}_{f_1} \cdot e^r - \underbrace{\left(\frac{\ln(\frac{S}{K} + (r - \frac{\sigma^2}{2}(T-i)))}{\sqrt{2}\sigma\sqrt{T-i}} \right)^{2k-1}}_{f_2} \right) > 0 \quad (16)$$

$T-(i+1) > 0$ yields $i < T-1$. (This condition is stronger than the other condition ($i < T$).)
To solve this inequality is very cumbersome. For this reason I calculated directly the F values on my notebook. In Appendix A you find the source code in C++ for the calculation of the instability values $F(i)$. The parameters (especially r and σ) are varied. At the end of this paper you find the F vs. i plots.

There is a numerical problem due to the limit of performance of my notebook. k cannot be extended to infinity. By trial and error, I found out that the limit is at $k_{max} = 30$. This is not a big problem due to factor $1/(k-1)!$ in eq. (16). The convergency velocity is very high. This is shown in figure (5) on page 11. After 3 k-steps the limit is reached. The error (in F units) is smaller than 10^{-4}.

4 Discussion of the $F(i)$ (instability function) plots

In figure (1) on page 7 the interest rate r equals 5% . The spot price of the stock S equals 120.0 and the strike/exercise price price K equals 80; units for K and S are USD or EUR etc. The expiry date is equal to 30 years. The volatility σ is varried. For high volatilities (greater than 40%) the value function of the option starts in the stability region. After 10 years all curves are in the instability region. At the expiry time $T = 30$ years the curves increases very fast. Stability close to T seems to be impossible.
In figure (2) on page 8 the volatility σ is equal to 50%. The interest rate r is varied. The higher the interest rate, the later is the transition from stability to instability. For $r = 25\%$ the order-chaos transition is at $t = 18$ years.
In figure (3) on page 9 the expiry time T is equal to 10 years. r is fixed and σ is varied. The stability-instability transition takes place at latest at 5 years for $\sigma = 50\%$. For $\sigma \leq 25\%$ the $F(i)$ curve is always in the stability region.
In figure (4) on page 10 there are two graphs. The graph at the bottom is a zoom into the region of the transition of stability and instability. The volatility σ is equal to 25% and $T = 30$ years. The interest rate is varied. The difference to the graphs before is that S is equal to 82.0 and K equals 80.0.
In figure (5) on page 11 it is shown (for one example) how fast the convergence of $F(k)$ is. Behind $k = 3$ the limiting value is reached.

Finally one can say that European call options cannot be traded without high risks over long period of some decades. After a few years(depending on interest rate r, volatility σ, and spot price S and strike/exercise price K) the instability region of the value function of the European call option is reached. For extreme high r and σ the transition of stability to instability can be at for instance at 18 years. But nevertheless at maturity of the European call option the value function is always in the instability region.

References

[1] Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets. World Scientific Publishing, Hagen Kleinert, 5th edition

[2] http://en.wikipedia.org/wiki/Black-Scholes google search word: "Black-Scholes Formula wiki", first result (March 29th 2011)

[3] Deterministic Chaos, Heinz Georg Schuster 3rd augemented edition, VCH

[4] Table of Integrals, Series, and Products, I.S. Gradshteyn and I.M. Ryzhik, 6th Edition, Academic Press

A C++ Source Code for the numerical calculation of $F(i)$ values

```
#include <cstdlib>
#include <iostream>
#include <math.h>
#include <fstream>

using namespace std;

int factorial(int n)
{
if (n==0) return 1;

int prod=1;

  for (int i=1;i<=n;i++)
  {
   prod = prod*i;
  }
  return prod;

}


int main(int argc, char *argv[])
{

    double r, sigma, T, K, S, t;
    int k,i;

    r=0.25 /*0.05*/;
    sigma=0.25;//0.25

    S=82.0;//120.0
    K=80.0;//80.0
    t=log(S/K);

    T=30;



    int k_max=30; // k_max>30 k-sum is not a number

    double sum,F;

    ofstream myfile;

    myfile.open("C:\\Users\\Sven\\Documents\\Projekt 7 Options\\r25_sigma_25_S82_K80.xls");



    for(i=0;i<=T-2;i++)
    {
    sum=F=0.0;

    for(k=1;k<=k_max;k++)
    {

    double f1= pow( ( t + ( r - 0.5*pow(sigma,2)*(T-(i+1)) ) / (sqrt(2)*sigma*sqrt( T- (i+1) ) ) ), (2*k-1));
    double f2= pow( ( t + ( r - 0.5*pow(sigma,2)*(T-i) ) / (sqrt(2)*sigma*sqrt(T-i) ) ), (2*k-1));

    double f = (pow(-1,(k+1))/(factorial(k-1)*(2*k-1)) )*( f1 * exp(r) - f2 );

    sum = sum + f;
```

```
    }
    F = sum;


    myfile << i << "\t" << F << endl;

     }

    myfile.close();
    return 0;


    system("PAUSE");
    return EXIT_SUCCESS;


}
```

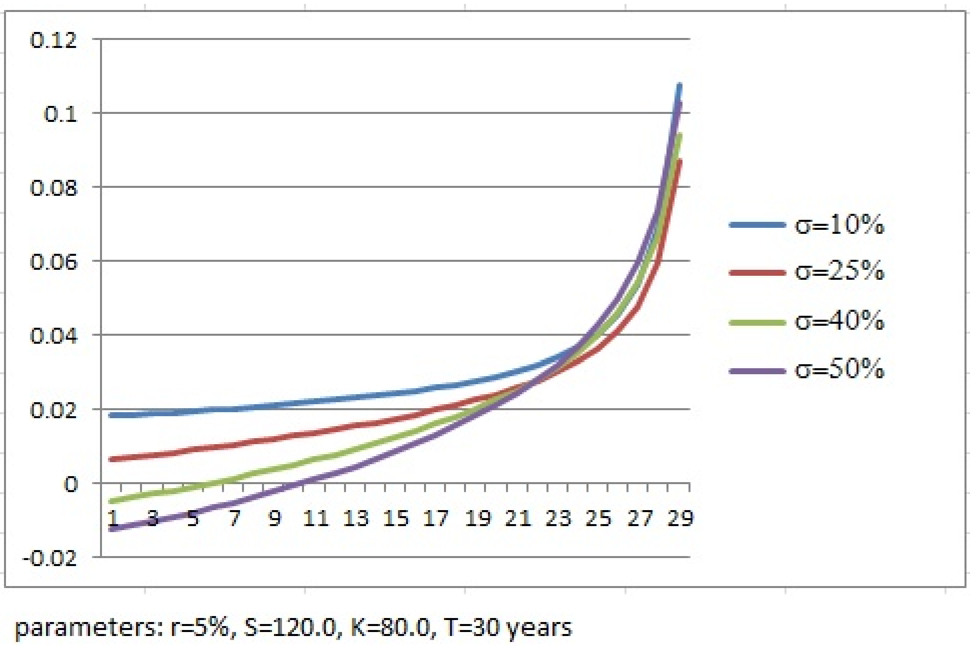

Figure 1: Instability F(i) vs. time i

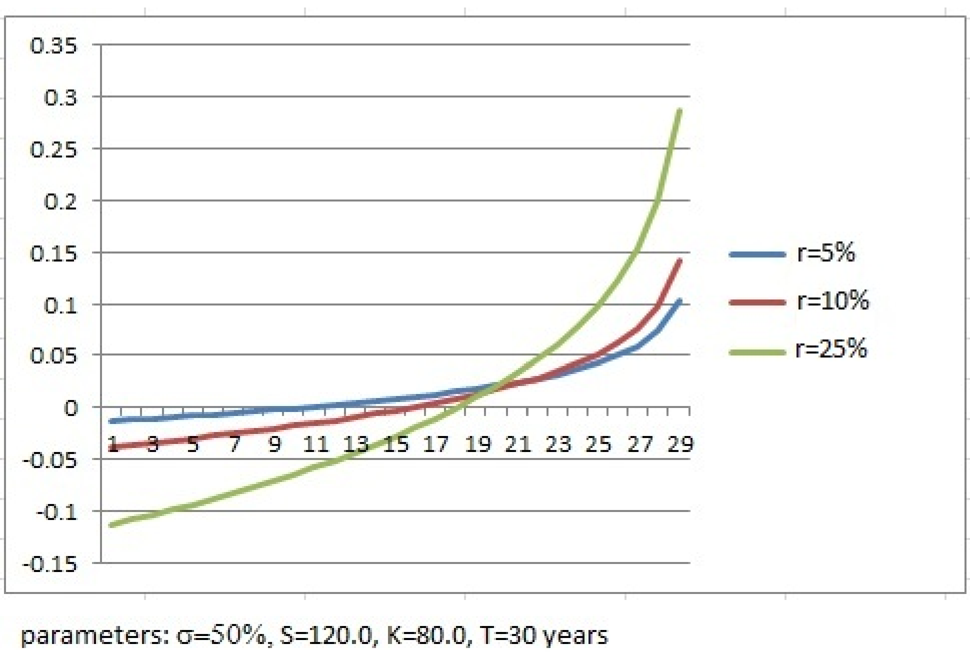

Figure 2: Instability F(i) vs. time i

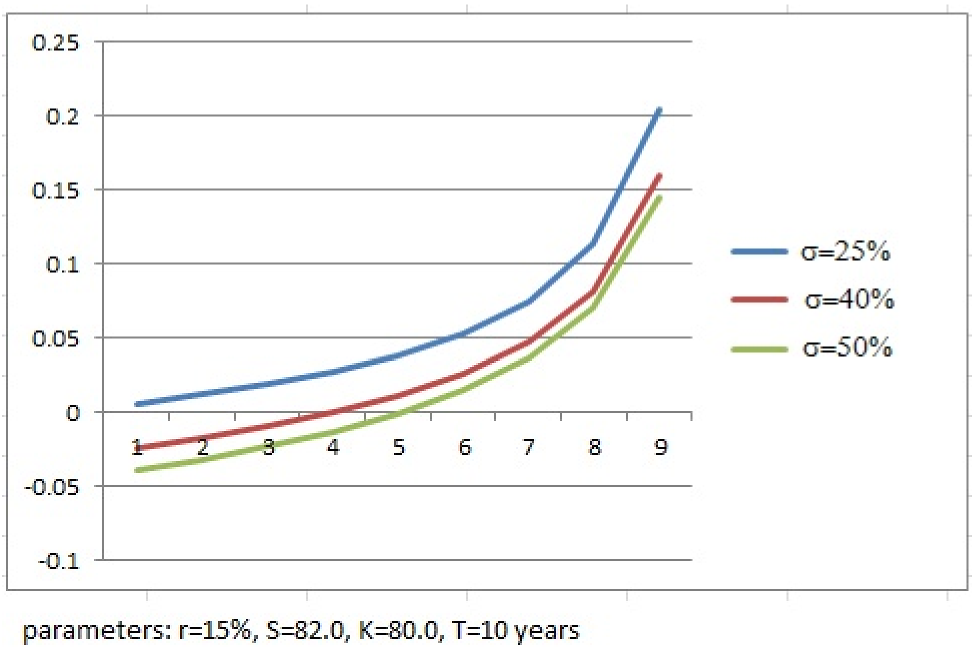

Figure 3: Instability F(i) vs. time i

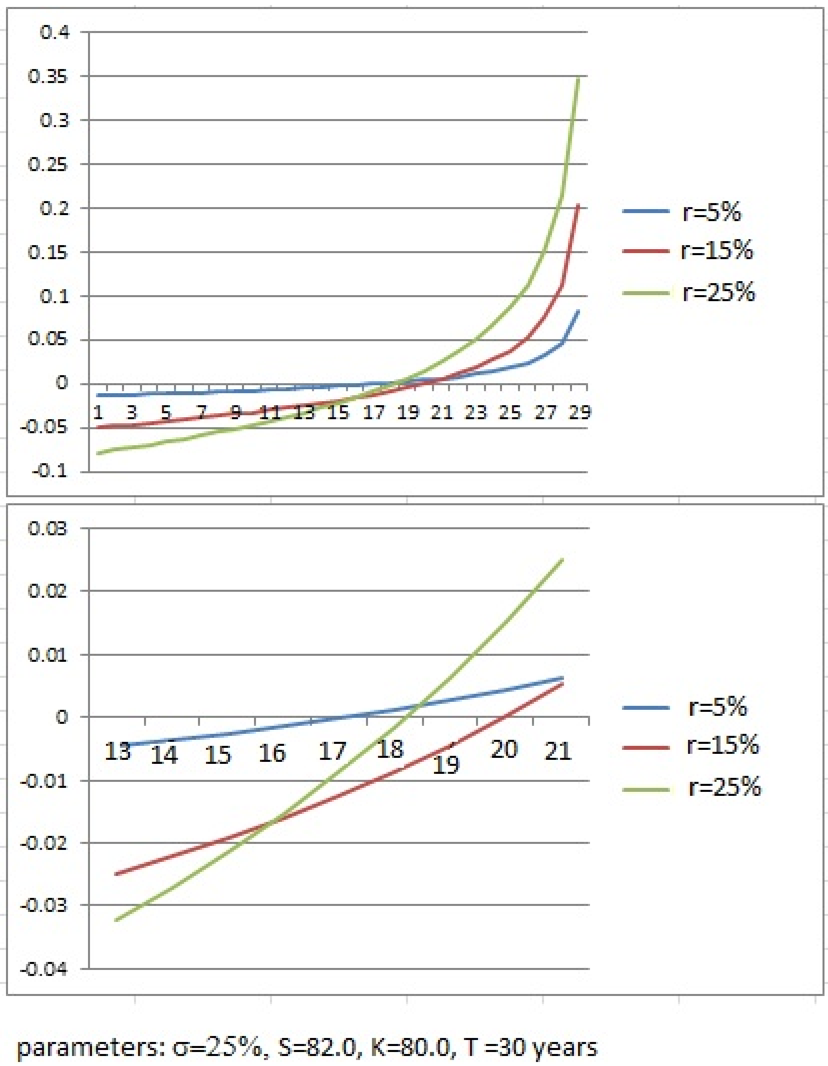

Figure 4: Instability F(i) vs. time i

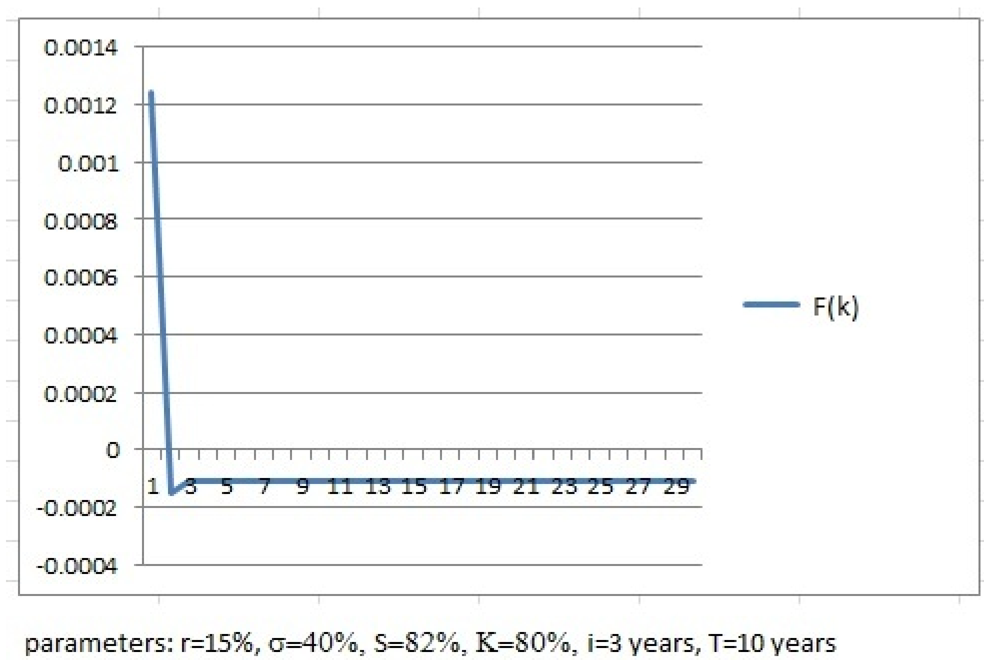

Figure 5: Instability F(k) vs. k